REAL ANALYSIS
TOPIC XII - THE BOLZANO-WEIERSTRAUSS THEOREM

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ABSTRACT. The Bolzano-Weierstrauss Theorem has many forms, and we will explore some of them.

Definition 1. Let X be a topological space. Let $A \subset X$ and $p \in X$. We say that p is an *accumulation point* of A if every deleted neighborhood of p intersects A.

Theorem 1. (Bolzano-Weierstrauss over $\mathbb{R}$, Version 1)
Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Let $T = \{x \in \mathbb{R} \mid (-\infty, x) \cap S \text{ is finite}\}$. We wish to show that T has a supremum, and that this supremum is an accumulation point of S.

Claim: The supremum of T exists.
If x_1 is a lower bound for S, then $(\infty, x_1) \cap S = \varnothing$, which is finite, so $x_1 \in T$. Thus T is nonempty. If x_2 is an upper bound for S and $x \geq x_2$, then $(\infty, x) \cap S = (\infty, x_2) \cap S = S$ is infinite, so $x \notin T$. Thus T is bounded above by x_2. By the Completeness Property of the real numbers, since T is a nonempty set which is bounded above, it has a supremum.

Let $p = \sup T$.

Claim: $(p - \epsilon, p + \epsilon) \cap S$ an infinite set, for every $\epsilon > 0$.
To see this, suppose not. Then $(p - \epsilon, p + \epsilon) \cap S$ is finite for some $\epsilon > 0$. By definition of p, the set $(-\infty, p - \epsilon) \cap S$ is finite, so the set $(-\infty, p + \epsilon) \cap S = ((-\infty, p - \epsilon) \cup \{p - \epsilon\} \cup (p - \epsilon, p + \epsilon)) \cap S$, being the union of finite sets, is finite. But then $p + \epsilon \in T$, so $p + \epsilon \leq \sup T = p$, contradicting that ϵ is positive. This contradiction shows that $(p - \epsilon, p + \epsilon) \cap S$ is infinite, for all $\epsilon > 0$.

Claim: p is an accumulation point of S
Every neighborhood of p contains an interval of the form $(p - \epsilon, p + \epsilon)$ for some $\epsilon > 0$. We have shown that each such set intersects S. Thus, p is an accumulation point of S. □

Next, we reformulate the statement of the theorem using sequences. Recall the following definition:

Date: January 24, 2019.

Theorem 2. (Bolzano-Weierstrauss over $\mathbb{R}$, Version 2)
Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. We have seen that every sequence in $\mathbb{R}$ has a monotone subsequence, and that every bounded monotonic sequence in $\mathbb{R}$ converges.

Let (x_n) be a bounded sequence in $\mathbb{R}$. Then (x_n) has a monotone subsequence, say (x_{n_k}). Clearly, (x_{n_k}) is also bounded, and thus is a bounded monotonic sequence in $\mathbb{R}$, which necessarily converges. □

Theorem 3. (Bolzano-Weierstrauss over $\mathbb{R}$, Version 3)
Every bounded sequence in $\mathbb{R}$ has a cluster point.

Proof. We have seen that a point is a cluster point of a sequence if and only if it is a subsequential limit.

Let (x_n) be a bounded sequence in $\mathbb{R}$. By Theorem 2, (x_n) has a convergent subsequence, say (x_{n_k}). Let $q = \lim_{k\to\infty} x_{n_k}$. Then q is a cluster point of (x_n). □

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